J Quant Spectrosc Radiat Transfer Vol 27, No 2 pp 141–148, 1982
Printed in Great Britain

0022–4073/82/020141–08$03 00/0
Pergamon Press Ltd

RADIATIVE TRANSFER IN FINITE INHOMOGENEOUS PLANE-PARALLEL ATMOSPHERES

R. D. M. GARCIA and C. E. SIEWERT
Departments of Nuclear Engineering and Mathematics, North Carolina State University, Raleigh, NC 27650, U S A

(*Received* 30 *March* 1981)

Abstract—The F_N method is used to deduce accurate numerical results for the exit distributions of radiation relevant to a finite, plane-parallel atmosphere with an exponentially varying albedo for single scattering

1 INTRODUCTION

We consider the radiative transfer problem defined by the equation of transfer

$$\mu\frac{\partial}{\partial\tau}I(\tau,\mu)+I(\tau,\mu)=\frac{1}{2}\omega(\tau)\int_{-1}^{1}I(\tau,\mu')\,d\mu' \tag{1}$$

and the boundary conditions

$$I(0,\mu)=F_1(\mu),\quad \mu>0, \tag{2a}$$

and

$$I(\tau_0,-\mu)=F_2(\mu),\quad \mu>0 \tag{2b}$$

Here $\tau\in[0,\tau_0]$ is the optical variable, μ is the direction cosine of the propagating radiation, and we consider an albedo for single scattering of the form

$$\omega(\tau)=\omega_0\,e^{-\tau/s} \tag{3}$$

where $0<\omega_0\leq 1$ and $s>0$. We assume $F_1(\mu)$ and $F_2(\mu)$ to be given, and we seek

$$A^*=\left(\int_0^1[F_1(\mu)+F_2(\mu)]\mu\,d\mu\right)^{-1}\int_0^1 I(0,-\mu)\mu\,d\mu, \tag{4a}$$

$$B^*=\left(\int_0^1[F_1(\mu)+F_2(\mu)]\mu\,d\mu\right)^{-1}\int_0^1 I(\tau_0,\mu)\mu\,d\mu, \tag{4b}$$

and the exit distributions $I(0,-\mu)$ and $I(\tau_0,\mu)$, $\mu>0$. In a recent paper[1] we used the elementary solutions reported by Mullikin and Siewert[2] to establish accurate numerical results for half-space applications, $\tau\in[0,\infty)$; here we extend the analysis and computational method in order to generate numerical results for a class of finite-slab problems, $\tau\in[0,\tau_0]$ Though there has been considerable discussion[3–7] concerning half-space problems and the appropriate completeness properties of the elementary solutions reported by Mullikin and Siewert,[2] the present study and a concurrent paper by Kelley[8] appear to be the first semi-analytical works that lead to numerical results for finite atmospheres with an exponentially varying albedo for single scattering.

2 ANALYSIS

We begin by expressing $I(\tau,\mu)$ in terms of the elementary solutions reported by Mullikin and Siewert,[2] i.e.

$$I(\tau,\mu)=\int_0^1 A(\nu)\Phi_\tau(\nu,\mu)\,e^{-\tau/\nu}\,d\nu+\int_S B(\eta)\Phi_\tau(-\eta,\mu)\,e^{\tau/\eta}\,d\eta, \tag{5}$$

where

$$\Phi_\tau(z,\mu)=\frac{1}{2}\omega_0 p(z)\,e^{-\tau/s}\left[Pv\left(\frac{1}{p(z)-\mu}\right)-2\tanh^{-1}p(z)\delta[p(z)-\mu]\right]+\delta(z-\mu), \tag{6}$$

$p(z)=sz/(s+z)$, $\eta\in S\Rightarrow\eta\in[0,s_1]$ or $\eta\in[0,s_1]\cup[s_2,1]$ if $s<1/2$, $s_1=s/(s+1)$ and $s_2=s/(1-s)$. In Ref 2 Mullikin and Siewert used the full-range orthogonality properties of the generalized functions $\Phi_\tau(z,\mu)$ and the representation for $I(\tau,\mu)$ given by Eq. (5) to deduce a system of singular integral equations for the desired surface distributions $I(0,-\mu)$ and $I(\tau_0,\mu)$, $\mu>0$. After changing variables, we write these equations as

$$\int_0^1 \mu X(0{:}\zeta,\mu)I(0,-\mu)\,d\mu+e^{-\tau_0/q(\zeta)}\int_0^1 \mu X(\tau_0{:}\zeta,-\mu)I(\tau_0,\mu)\,d\mu=T_1(\zeta), \tag{7a}$$

for $\zeta\in S$, and

$$\int_0^1 \mu Y(\tau_0{:}\xi,\mu)I(\tau_0,\mu)\,d\mu+e^{-\tau_0/p(\xi)}\int_0^1 \mu Y(0{:}\xi,-\mu)I(0,-\mu)\,d\mu=T_2(\xi), \tag{7b}$$

for $\xi\in(0,1)$. Here the known terms are

$$T_1(\zeta)=\int_0^1 \mu X(0{:}\zeta,-\mu)F_1(\mu)\,d\mu+e^{-\tau_0/q(\zeta)}\int_0^1 \mu X(\tau_0{:}\zeta,\mu)F_2(\mu)\,d\mu \tag{8a}$$

and

$$T_2(\xi)=\int_0^1 \mu Y(\tau_0{:}\xi,-\mu)F_2(\mu)\,d\mu+e^{-\tau_0/p(\xi)}\int_0^1 \mu Y(0{:}\xi,\mu)F_1(\mu)\,d\mu. \tag{8b}$$

In addition

$$X(\tau{:}\zeta,\mu)=\frac{1}{2}\omega_0\zeta\,e^{-\tau/s}\left[Pv\left(\frac{1}{\zeta-\mu}\right)-2\tanh^{-1}\zeta\delta(\zeta-\mu)\right]+\delta[q(\zeta)-\mu] \tag{9a}$$

and

$$Y(\tau{:}\xi,\mu)=\frac{1}{2}\omega_0\xi\,e^{-\tau/s}\left[Pv\left(\frac{1}{\xi-\mu}\right)-2\tanh^{-1}\xi\delta(\xi-\mu)\right]+\delta[p(\xi)-\mu] \tag{9b}$$

where $q(\zeta)=s\zeta/(s-\zeta)$ If we now consider Eq. (5) for $\tau=0$ and $\tau=\tau_0$, we can deduce the following representations for the exit distributions $I(0,-\mu)$ and $I(\tau_0,\mu)$, $\mu>0$:

$$I(0,-\mu)=F_2(\mu)\,e^{-\tau_0/\mu}+\frac{1}{2}\omega_0 L(\mu) \tag{10a}$$

and

$$I(\tau_0,\mu)=F_1(\mu)\,e^{-\tau_0/\mu}+\frac{1}{2}\omega_0 R(\mu) \tag{10b}$$

where

$$L(\mu)=\int_S \eta D(\eta)S(\eta,\mu)\,d\eta+\int_0^1 \nu E(\nu)C(\nu,\mu)\,d\nu \tag{11a}$$

and

$$R(\mu)=\int_S \eta D(\eta)C(\eta,\mu)\,d\eta+\int_0^1 \nu E(\nu)S(\nu,\mu)\,d\nu \tag{11b}$$

In addition

$$S(x, y) = \frac{1 - \exp\left[-\tau_0\left(\frac{1}{x} + \frac{1}{y}\right)\right]}{x + y} \tag{12a}$$

and

$$C(x, y) = \frac{e^{-\tau_0/x} - e^{-\tau_0/y}}{x - y}. \tag{12b}$$

On substituting Eqs. (10) into Eqs. (7), we find

$$e^{-\tau_0/s}\int_0^1 \mu X(0:\zeta, \mu)L(\mu)\,d\mu + e^{-\tau_0/\zeta}\int_0^1 \mu X(\tau_0:\zeta, -\mu)R(\mu)\,d\mu = K_1(\zeta), \tag{13a}$$

for $\zeta \in S$, and

$$\int_0^1 \mu Y(\tau_0:\xi, \mu)R(\mu)\,d\mu + e^{-\tau_0/p(\xi)}\int_0^1 \mu Y(0:\xi, -\mu)L(\mu)\,d\mu = K_2(\xi), \tag{13b}$$

for $\xi \in (0, 1)$. Here

$$K_1(\zeta) = \zeta\, e^{-\tau_0/s}\int_0^1 \mu[F_1(\mu)S(\mu, \zeta) + F_2(\mu)C(\mu, \zeta)]\,d\mu \tag{14a}$$

and

$$K_2(\xi) = \xi\, e^{-\tau_0/s}\int_0^1 \mu[F_1(\mu)C(\mu, \xi) + F_2(\mu)S(\mu, \xi)]\,d\mu. \tag{14b}$$

Equations (11) suggest the approximations

$$L(\mu) = \sum_{\alpha=0}^{N}[a_\alpha\zeta_\alpha S(\zeta_\alpha, \mu) + b_\alpha\xi_\alpha C(\xi_\alpha, \mu)] \tag{15a}$$

and

$$R(\mu) = \sum_{\alpha=0}^{N}[a_\alpha\zeta_\alpha C(\zeta_\alpha, \mu) + b_\alpha\xi_\alpha S(\xi_\alpha, \mu)] \tag{15b}$$

where $\{\zeta_\alpha\}$ is a collection of points contained in S, $\{\xi_\alpha\}$ is a collection of points contained in the interval (0, 1) and the constants $\{a_\alpha\}$ and $\{b_\alpha\}$ are to be determined If we now substitute Eqs. (15) into Eqs (13) we find, for $\zeta \in S$,

$$\sum_{\alpha=0}^{N}\left\{a_\alpha\left[sr(\zeta, \zeta_\alpha)C[s, r(\zeta, \zeta_\alpha)] + \frac{1}{2}\omega(\tau_0)\Gamma(\zeta_\alpha, \zeta)\right] + b_\alpha\left[p(\xi_\alpha)\zeta C[p(\xi_\alpha), \zeta] + \frac{1}{2}\omega(\tau_0)\Delta(\xi_\alpha, \zeta)\right]\right\} = K_1(\zeta) \tag{16a}$$

and for $\xi \in (0, 1)$,

$$\sum_{\alpha=0}^{N}\left\{a_\alpha\left[\zeta_\alpha p(\xi)C[\zeta_\alpha, p(\xi)] + \frac{1}{2}\omega(\tau_0)\Delta(\zeta_\alpha, \xi)\right] + b_\alpha\left[\xi_\alpha p(\xi)S[\xi_\alpha, p(\xi)] + \frac{1}{2}\omega(\tau_0)\Gamma(\xi_\alpha, \xi)\right]\right\} = K_2(\xi), \tag{16b}$$

where

$$r(x, y) = \frac{xy}{x + y}, \tag{17}$$

$$\Gamma(x, y) = xy[U(x, y) + U(y, x)] \tag{18a}$$

and

$$\Delta(x, y) = xy[V(x, y) + V(y, x) - W(x, y)]. \tag{18b}$$

Here we have used the definitions

$$U(x, y) = e^{-\tau_0/x}\int_0^1 \mu C(y, \mu)\frac{d\mu}{\mu + x} - x\log(1 + 1/x)S(x, y), \tag{19}$$

$$V(x, y) = \left(\frac{1}{x - y}\right)e^{-\tau_0/y}\left[1 + x\log(1 + 1/x) + e^{-\tau_0/x}\int_0^1 \mu\, e^{-\tau_0/\mu}\frac{d\mu}{\mu + x}\right], \tag{20}$$

and

$$W(x, y) = \left(\frac{1}{x - y}\right)[C_1(x) - C_1(y)], \tag{21}$$

where

$$C_1(z) = \int_0^1 \mu C(z, \mu)\,d\mu. \tag{22}$$

If we consider Eq (16a) for $N+1$ values of $\zeta \in S$ and Eq (16b) for $N+1$ values of $\xi \in (0, 1)$ we obtain a system of $2(N+1)$ linear algebraic equations that can be solved to yield the desired constants $\{a_\alpha\}$ and $\{b_\alpha\}$ We note that in the limiting case of $\tau_0 \to \infty$, Eq (16b) can be disregarded, while Eq. (16a) for $\zeta \in (0, s_1)$ reduces to the half-space result reported in Ref. 1

3 NUMERICAL RESULTS

In order to evaluate the foregoing analysis we have investigated two specific classes of problems. We first took $F_1(\mu) = 1$ and $F_2(\mu) = 0$ and solved the system of $2(N+1)$ linear algebraic equations resulting from Eqs. (16) to find the desired constants $\{a_\alpha\}$ and $\{b_\alpha\}$ We list in Tables 1 and 2 our converged results for the exit distributions

$$I(0, -\mu) = \frac{1}{2}\omega_0\sum_{\alpha=0}^{N}[a_\alpha\zeta_\alpha S(\zeta_\alpha, \mu) + b_\alpha\xi_\alpha C(\xi_\alpha, \mu)] \tag{23a}$$

and

$$I(\tau_0, \mu) = e^{-\tau_0/\mu} + \frac{1}{2}\omega_0\sum_{\alpha=0}^{N}[a_\alpha\zeta_\alpha C(\zeta_\alpha, \mu) + b_\alpha\xi_\alpha S(\xi_\alpha, \mu)] \tag{23b}$$

Table 1 The exit distribution $I(0, -\mu)$ for $\omega_0 = 1$, $\tau_0 = 5$, $F_1(\mu) = 1$ and $F_2(\mu) = 0$

μ	$s = 1$	$s = 10$	$s = 10^2$	$s = 10^3$	$s = \infty$
0.05	0.58966	0.76081	0.86400	0.89361	0.89780
0.1	0.53112	0.73398	0.85028	0.88319	0.88784
0.2	0.44328	0.68632	0.82523	0.86410	0.86958
0.3	0.38031	0.64418	0.80181	0.84606	0.85230
0.4	0.33296	0.60647	0.77946	0.82857	0.83550
0.5	0.29609	0.57257	0.75799	0.81143	0.81900
0.6	0.26656	0.54197	0.73730	0.79455	0.80268
0.7	0.24239	0.51427	0.71733	0.77788	0.78649
0.8	0.22223	0.48910	0.69805	0.76139	0.77043
0.9	0.20517	0.46615	0.67943	0.74510	0.75450
1.0	0.19055	0.44517	0.66146	0.72904	0.73872

Table 2 The exit distribution $I(\tau_0, \mu)$ for $\omega_0 = 1$, $\tau_0 = 5$, $F_1(\mu) = 1$ and $F_2(\mu) = 0$

μ	$s = 1$	$s = 10$	$s = 10^2$	$s = 10^3$	$s = \infty$
0 05	0 6075(−5)	0 58031(−2)	0.62883(−1)	0.96845(−1)	0 10220
0 1	0 69252(−5)	0 63702(−2)	0 69024(−1)	0 10629	0 11216
0 2	0 96423(−5)	0 76183(−2)	0 80567(−1)	0 12363	0.13042
0 3	0 16234(−4)	0 91482(−2)	0 91918(−1)	0 14012	0 14770
0 4	0 43858(−4)	0.11119(−1)	0 10342	0 15622	0 16450
0 5	0.16937(−3)	0 13725(−1)	0 11523	0 17211	0 18100
0 6	0 57347(−3)	0 17183(−1)	0 12744	0 18790	0 19732
0 7	0 15128(−2)	0 21680(−1)	0 14010	0 20364	0 21351
0.8	0 32437(−2)	0 27331(−1)	0 15319	0.21933	0 22957
0 9	0 59604(−2)	0 34166(−1)	0 16667	0 23496	0 24550
1 0	0 97712(−2)	0 42142(−1)	0 18047	0 25049	0 26128

Table 3 A^* for $F_1(\mu) = 1$ and $F_2(\mu) = 0$

ω_0	s	$\tau_0 = 0\ 1$	$\tau_0 = 1$	$\tau_0 = 5$	$\tau_0 = 10$
	1	0 530284(−1)	0 152071	0.155240	0 155240
	10	0 554877(−1)	0 211600	0 235414	0 235418
0 7	10^2	0 557431(−1)	0 220966	0 254017	0 254044
	10^3	0 557687(−1)	0 221959	0 256264	0.256300
	∞	0 557716(−1)	0 222070	0 256519	0.256557
	1	0 706233(−1)	0 218920	0 224315	0 224315
	10	0 740329(−1)	0 330211	0.395420	0 395445
0 9	10^2	0 743877(−1)	0 350287	0 463707	0 464543
	10^3	0 744233(−1)	0 352468	0 474979	0 476532
	∞	0 744273(−1)	0 352712	0 476338	0 478016
	1	0 799031(−1)	0 258891	0 265892	0 265892
	10	0 838411(−1)	0 412506	0 531182	0 531268
1 0	10^2	0 842514(−1)	0 442863	0 725972	0 741935
	10^3	0 842925(−1)	0 446217	0.784073	0 854489
	∞	0 842971(−1)	0 446594	0.792343	0 883255

Table 4 B^* for $F_1(\mu) = 1$ and $F_2(\mu) = 0$

ω_0	s	$\tau_0 = 0\ 1$	$\tau_0 = 1$	$\tau_0 = 5$	$\tau_0 = 10$
	1	0 884557	0 293658	0 233529(−2)	0 918404(−5)
	10	0 887231	0 358263	0 647245(−2)	0 325450(−4)
0 7	10^2	0 887509	0 369812	0 113268(−1)	0 139581(−3)
	10^3	0 887537	0 371056	0 122743(−1)	0 186945(−3)
	∞	0 887540	0 371195	0 123892(−1)	0 193749(−3)
	1	0 901829	0.328284	0 261169(−2)	0 101773(−4)
	10	0 905526	0.447381	0.136607(−1)	0 800151(−4)
0 9	10^2	0 905912	0 471755	0 431419(−1)	0 164786(−2)
	10^3	0 905950	0 474444	0.522190(−1)	0 349561(−2)
	∞	0.905955	0 474746	0 534214(−1)	0 385558(−2)
	1	0 910943	0.349433	0.278246(−2)	0 107906(−4)
	10	0 915208	0 512428	0 228739(−1)	0 148970(−3)
1 0	10^2	0 915653	0 548858	0 136939	0 157519(−1)
	10^3	0 915698	0 552946	0 198147	0 868899(−1)
	∞	0 915703	0 553406	0 207657	0 116745

In Tables 3 and 4 we report our results for

$$A^* = \omega_0 \sum_{\alpha=0}^{N} [a_\alpha \zeta_\alpha S_1(\zeta_\alpha) + b_\alpha \xi_\alpha C_1(\xi_\alpha)] \tag{24a}$$

and

$$B^* = 2E_3(\tau_0) + \omega_0 \sum_{\alpha=0}^{N} [a_\alpha \zeta_\alpha C_1(\zeta_\alpha) + b_\alpha \xi_\alpha S_1(\xi_\alpha)]. \tag{24b}$$

Here

$$S_1(x) = \int_0^1 \mu S(x, \mu)\, d\mu \tag{25}$$

and $E_3(x)$ denotes an exponential integral function. For our second study we took $F_1(\mu) = \delta(\mu - \mu_0)$, $\mu_0 = i/10$, $i = 1, 2, \ldots, 10$, and $F_2(\mu) = 0$. In Tables 5 and 6 we list, for $\mu_0 = 0.9$, converged results for the exit distributions $I(0, -\mu)$ and $I^*(\tau_0, \mu) = I(\tau_0, \mu) - \delta(\mu - \mu_0)\exp(-\tau_0/\mu)$. In Tables 7 and 8 we list our results for

$$A^* = \frac{1}{2\mu_0}\omega_0 \sum_{\alpha=0}^{N} [a_\alpha \zeta_\alpha S_1(\zeta_\alpha) + b_\alpha \xi_\alpha C_1(\xi_\alpha)] \tag{26a}$$

Table 5. The exit distribution $I(0, -\mu)$ for $\omega_0 = 1$, $\tau_0 = 5$, $F_1(\mu) = \delta(\mu - 0.9)$ and $F_2(\mu) = 0$

μ	$s = 1$	$s = 10$	$s = 10^2$	$s = 10^3$	$s = \infty$
0.05	0.69801	0.98398	0.11842(+1)	0.12441(+1)	0.12526(+1)
0.1	0.65651	0.99343	0.12182(+1)	0.12846(+1)	0.12940(+1)
0.2	0.57637	0.98472	0.12531(+1)	0.13315(+1)	0.13426(+1)
0.3	0.50955	0.95989	0.12655(+1)	0.13549(+1)	0.13676(+1)
0.4	0.45523	0.92849	0.12655(+1)	0.13649(+1)	0.13790(+1)
0.5	0.41078	0.89472	0.12576(+1)	0.13660(+1)	0.13814(+1)
0.6	0.37396	0.86064	0.12446(+1)	0.13609(+1)	0.13775(+1)
0.7	0.34304	0.82733	0.12280(+1)	0.13512(+1)	0.13688(+1)
0.8	0.31676	0.79531	0.12089(+1)	0.13381(+1)	0.13566(+1)
0.9	0.29416	0.76486	0.11883(+1)	0.13223(+1)	0.13416(+1)
1.0	0.27454	0.73605	0.11665(+1)	0.13046(+1)	0.13245(+1)

Table 6. The exit distribution $I^*(\tau_0, \mu)$ for $\omega_0 = 1$, $\tau_0 = 5$, $F_1(\mu) = \delta(\mu - 0.9)$ and $F_2(\mu) = 0$

μ	$s = 1$	$s = 10$	$s = 10^2$	$s = 10^3$	$s = \infty$
0.05	0.18678(−4)	0.13216(−1)	0.13460	0.20601	0.21726
0.1	0.21226(−4)	0.14498(−1)	0.14772	0.22608	0.23842
0.2	0.29214(−4)	0.17307(−1)	0.17237	0.26291	0.27716
0.3	0.47082(−4)	0.20723(−1)	0.19656	0.29788	0.31378
0.4	0.10218(−3)	0.25054(−1)	0.22097	0.33190	0.34927
0.5	0.26855(−3)	0.30578(−1)	0.24579	0.36525	0.38390
0.6	0.65035(−3)	0.37422(−1)	0.27089	0.39779	0.41754
0.7	0.13276(−2)	0.45484(−1)	0.29586	0.42916	0.44984
0.8	0.23249(−2)	0.54487(−1)	0.32019	0.45887	0.48032
0.9	0.36161(−2)	0.64073(−1)	0.34339	0.48649	0.50858
1.0	0.51444(−2)	0.73894(−1)	0.36507	0.51173	0.53431

Table 7. A^* for $F_1(\mu) = \delta(\mu - 0.9)$ and $F_2(\mu) = 0$

ω_0	s	$\tau_0 = 0.1$	$\tau_0 = 1$	$\tau_0 = 5$	$\tau_0 = 10$
	1	0.330938(−1)	0.115005	0.118853	0.118853
	10	0.346827(−1)	0.168417	0.196979	0.196985
0.7	10^2	0.348478(−1)	0.177022	0.216745	0.216786
	10^3	0.348644(−1)	0.177937	0.219185	0.219239
	∞	0.348662(−1)	0.178039	0.219464	0.219519
	1	0.440808(−1)	0.166283	0.172617	0.172617
	10	0.462818(−1)	0.265719	0.340243	0.340278
0.9	10^2	0.465110(−1)	0.284075	0.414247	0.415321
	10^3	0.465340(−1)	0.286074	0.426863	0.428844
	∞	0.465366(−1)	0.286298	0.428393	0.430530
	1	0.498765(−1)	0.197101	0.205174	0.205174
	10	0.524177(−1)	0.334012	0.466152	0.466262
1.0	10^2	0.526825(−1)	0.361677	0.679431	0.698436
	10^3	0.527091(−1)	0.364743	0.745105	0.828402
	∞	0.527121(−1)	0.365087	0.754496	0.861963

Table 8 B^* for $F_1(\mu) = \delta(\mu - 0.9)$ and $F_2(\mu) = 0$

ω_0	s	$\tau_0 = 0.1$	$\tau_0 = 1$	$\tau_0 = 5$	$\tau_0 = 10$
	1	0.927586	0.394446	0.444660(−2)	0.169928(−4)
	10	0.929322	0.457548	0.101850(−1)	0.515608(−4)
0.7	10^2	0.929503	0.468911	0.169413(−1)	0.210501(−3)
	10^3	0.929521	0.470135	0.182531(−1)	0.280169(−3)
	∞	0.929523	0.470272	0.184121(−1)	0.290159(−3)
	1	0.938462	0.424179	0.470936(−2)	0.179231(−4)
	10	0.940862	0.537932	0.187395(−1)	0.110683(−3)
0.9	10^2	0.941112	0.561324	0.554371(−1)	0.212273(−2)
	10^3	0.941137	0.563906	0.666423(−1)	0.446877(−2)
	∞	0.941140	0.564195	0.681249(−1)	0.492480(−2)
	1	0.944201	0.442193	0.486873(−2)	0.184882(−4)
	10	0.946967	0.596084	0.291545(−1)	0.191149(−3)
1.0	10^2	0.947256	0.630603	0.163099	0.187701(−1)
	10^3	0.947285	0.634477	0.234432	0.102816
	∞	0.947288	0.634913	0.245504	0.138037

and

$$B^* = e^{-\tau_0/\mu_0} + \frac{1}{2\mu_0}\omega_0 \sum_{\alpha=0}^{N} [a_\alpha \zeta_\alpha C_1(\zeta_\alpha) + b_\alpha \xi_\alpha S_1(\xi_\alpha)]. \tag{26b}$$

In order to simplify matters we have chosen the collocation points for Eqs. (16) to be the same as the basis points, i.e. $\{\zeta_\alpha\}$ and $\{\xi_\alpha\}$. For a particular F_N approximation these were chosen according to the scheme[9]

$$\xi_\alpha = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{2\alpha+1}{2N+2}\pi\right) \tag{27a}$$

and

$$\zeta_\alpha = s_1 \xi_\alpha, \quad \alpha = 0, 1, 2 \ldots N, \tag{27b}$$

when $s < \infty$. For a homogeneous atmosphere, $s = \infty$, we note that $\zeta_\alpha = \xi_\alpha$; we therefore used the scheme

$$\xi_0 = \nu_0, \text{ all } N, \tag{28a}$$

and

$$\xi_\alpha = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{2\alpha-1}{2N}\pi\right), \quad \alpha = 1, 2 \ldots N, \quad N \neq 0, \tag{28b}$$

where ν_0 is the discrete eigenvalue appropriate to the homogeneous case. Using a 40-point Gaussian quadrature set to evaluate all required integrals, we have deduced our converged results with N typically less than 14. We believe that these results are accurate to within ± 1 in the last figure shown.

From the study of cases in addition to those reported here, we have found for large values of τ_0 and s and $\omega_0 \geq 0.9$ that a collocation scheme that includes the effect of ν_0, as discussed in Ref. 1, can be used to improve the rate of convergence as N is increased. However for very large τ_0 and s, say $\tau_0 > 20$ and $s > 10^6$, we have discovered a deterioration in the accuracy of our method, especially for the transmitted quantities $I^*(\tau_0, \mu)$ and B^*. We attribute this loss of accuracy to the fact that Eqs. (16) become almost linearly dependent and thus numerical difficulties are encountered when we attempt to invert the resulting ill-conditioned matrix. Possible ways to avoid these numerical difficulties are the subject of continuing work on this problem.

Acknowledgements—The authors are grateful to D W Decker and C T Kelley for several helpful discussions related to this work, which was supported in part by the U S National Science Foundation through grants INT-8000517 and CPE 8016775 In addition, one of the authors (R D.M G) wishes to note with thanks the financial support of Comissão Nacional de Energia Nuclear and Instituto de Pesquisas Energéticas e Nucleares, both of Brazil

REFERENCES

1 R D M Garcia and C E Siewert, *JQSRT* **25**, 277 (1981)
2 T W Mullikin and C E Siewert, *Annals Nucl Energy* 7, 205 (1980)
3 G C Pomraning and E W Larsen, *J Math Phys* **21**, 1603 (1980)
4 E W Larsen, G C Pomraning, and V C Badham, *J Math Phys* **21**, 2448 (1980)
5 E W Larsen and T W Mullikin, *J Math Phys* **22**, 856 (1981)
6 C T Kelley and T W Mullikin, *J Integral Equations* (in press)
7 E W Larsen, *J Math Phys* **22**, 1463 (1981)
8 C T Kelley, *J Integral Equations* (in press)
9 R D M Garcia and C E Siewert, *Nucl Sci Eng* **78**, 315 (1981)