Duration Calculus
Introduction

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Overview

- Background
- Short introduction
- Decidability results
- Undecidability results

Background

- Provable Correct Systems (ProCoS, ESPRIT BRA 3104)
  Bjørner Langmaack Hoare Olderog
- Project case study: Gas Burner Sørensen Ravn Rischel

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- Logical Calculi, Applications, Mechanical Support
- Duration Calculus: A formal approach to real-time systems
  Zhou Chaochen and Michael R. Hansen
  Springer 2004

Gas Burner example: Requirements

State variables modelling Gas and Flame:

$$\mathrm{G}, \mathrm{F} : \mathbb{T}\text{ime} \rightarrow \{0, 1\}$$

State expression modelling that gas is Leaking

$$\mathrm{L} \mathrel{\widehat{=}} \mathrm{G} \land \neg \mathrm{F}$$

Requirement

- Gas must at most be leaking 1/20 of the elapsed time

$$(e - b) \geq 60\,\mathrm{s} \;\Rightarrow\; 20 \int_b^e \mathrm{L}(t)dt \leq (e - b)$$

Gas Burner example: Design decisions

- Leaks are detectable and stoppable within 1s:

$$\forall c, d : b \leq c < d \leq e.(\mathrm{L}[c, d] \Rightarrow (d - c) \leq 1\,\mathrm{s})$$

where

$$P[c, d] \;\widehat{=}\; \int_c^d P(t) = (d - c) > 0$$

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- At least 30s between leaks:

  $$\forall c, d, r, s : b \le c < r < s < d \le e. \\ (\mathrm{L}[c, r] \wedge \neg \mathrm{L}[r, s] \wedge \mathrm{L}[s, d]) \;\Rightarrow\; (s - r) \ge 30\,\mathrm{s}$$

Interval Logic [Halpern Moszkowski Manna 83]

Terms: $\theta \ ::= \ x \mid v \mid \theta_1 + \theta_n \mid \ \ldots$ Temporal Variable

Formulas: $\phi \ ::= \ \theta_1 = \theta_n \mid \neg\phi \mid \phi \vee \psi \mid \phi \frown \psi \mid (\exists x)\phi \mid \ \ldots$ chop

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$\phi : \mathbb{I}\text{ntv} \rightarrow \{\text{tt,ff}\}$

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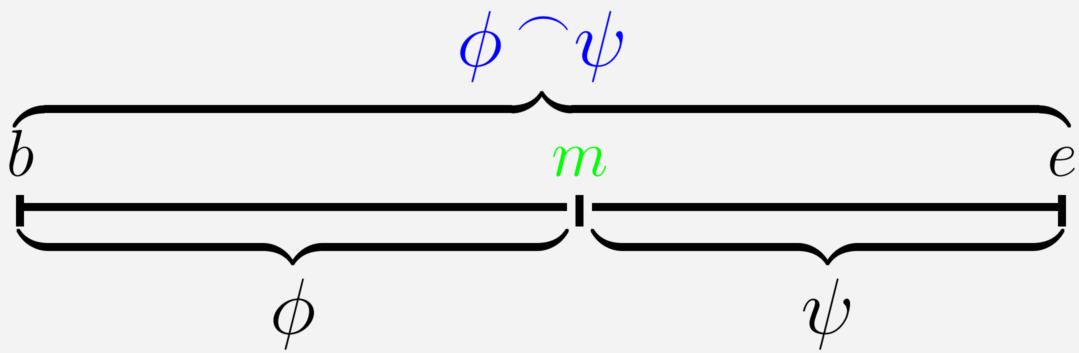
Chop:



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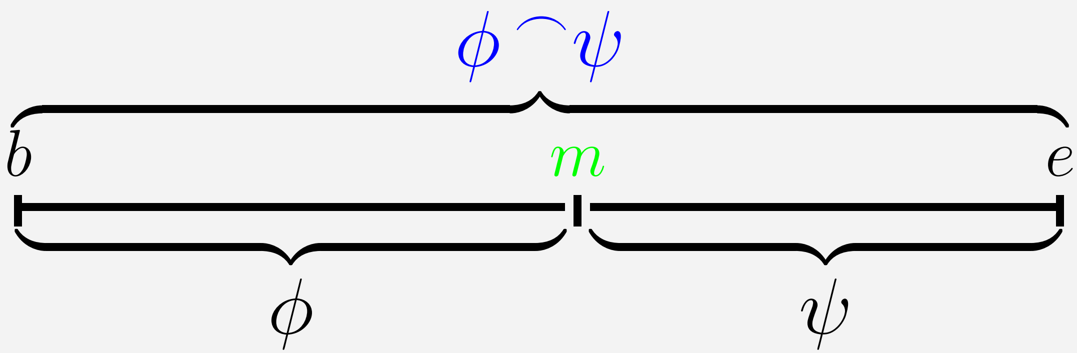

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In DC: $\mathbb{I}\text{ntv} = \{ \ [a, b] \mid a, b \in \mathbb{R} \wedge a \leq b\}$

Duration Calculus [Zhou Hoare Ravn 91]

- State variables $P: \mathbb{T}\text{ime} \rightarrow \{0,1\}$ Finite Variablilty
- State expressions $S ::= 0 \mid 1 \mid P \mid \neg S \mid S_1 \vee S_2$

  $S: \mathbb{T}\text{ime} \rightarrow \{0,1\}$ pointwise defined

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- Durations $\int S : \mathbb{I}\text{ntv} \longrightarrow \mathbb{R}$ defined on $[b, e]$ by

  $$\int_b^e S(t)dt$$

  – Temporal variables with a structure

Example: Gas Burner

Requirement

$$\ell \geq 60 \;\Rightarrow\; 20\int \mathrm{L} \leq \ell$$

Design decisions

$$D_1 \;\widehat{=}\; \Box(\lceil\!\lceil \mathrm{L} \rceil\!\rceil \Rightarrow \ell \leq 1)$$

$$D_2 \;\widehat{=}\; \Box((\lceil\!\lceil \mathrm{L} \rceil\!\rceil \frown \lceil\!\lceil \neg\mathrm{L} \rceil\!\rceil \frown \lceil\!\lceil \mathrm{L} \rceil\!\rceil) \Rightarrow \ell \geq 30)$$

where ℓ denotes the *length* of the interval, and

$\Diamond\phi$	$\widehat{=}$	true $\frown \phi \frown$ true	"for some sub-interval: ϕ"
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succinct formulation — no interval endpoints

Correctness — informal interval reasoning

We must establish: $(D_1 \ \wedge \ D_2) \Rightarrow \ \ell \geq 60 \ \Rightarrow \ 20 \int \mathrm{L} \leq \ell$

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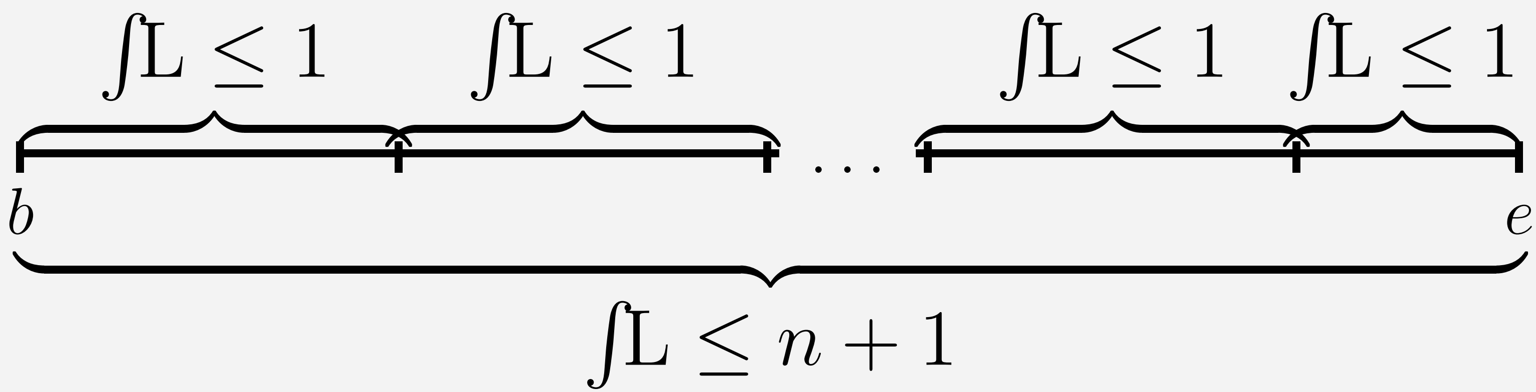

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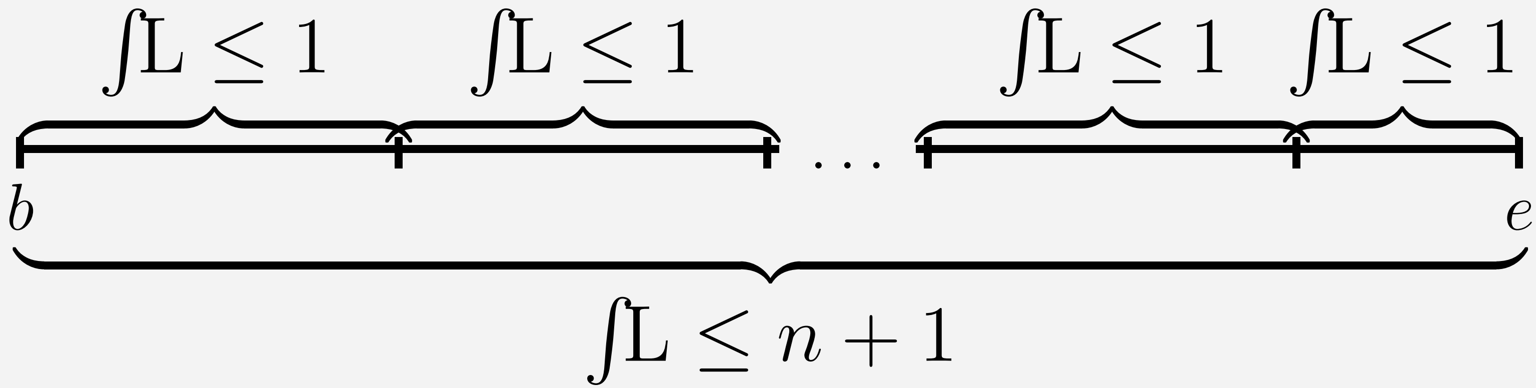

Since $n \geq 2 \Rightarrow 20 \cdot (n+1) \leq 30 \cdot n$ we have

$$(D_1 \;\wedge\; D_2) \Rightarrow \;\ell \geq 60 \;\Rightarrow\; 20\int\mathrm{L} \leq \ell$$

Decidability

[Zhou Hansen Sestoft 93]

Restricted Duration Calculus :

- $\llbracket S \rrbracket$
- $\neg\phi,\ \phi \vee \psi,\ \phi ^\frown \psi$

Satisfiability is reduced to emptiness of regular languages

Both for discrete and continuous time

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Even small extensions give undecidable subsets

RDC_1 (Cont. time)	RDC_2	RDC_3
• $\ell = r,\ \llbracket S \rrbracket$ • $\neg\phi,\ \phi\vee\psi, \phi\frown\psi$	• $\int S_1 = \int S_2$ • $\neg\phi,\ \phi\vee\psi, \phi\frown\psi$	• $\ell = x,\ \llbracket S \rrbracket$ • $\neg\phi,\ \phi\vee\psi, \phi\frown\psi,\ (\exists x)\phi$

Discrete-Time Duration Calculus

- For an interpretation

$$\mathcal{I} \; : \; SVar \longrightarrow (\mathbb{T}\text{ime} \longrightarrow \{0, 1\})$$

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- We consider only *discrete intervals*

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where $b, e \in \mathbb{N}$.

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$$[b,e] \in \mathbb{I}\text{ntv}$$

where $b, e \in \mathbb{N}$.

- The semantics of chop is

$$\mathcal{I},[b,e] \models \phi \frown \psi \;\text{ iff }\; \left\{ \begin{array}{l} \mathcal{I},[b,m] \models \phi \text{ and } \mathcal{I},[m,e] \models \psi, \\ \text{for some } m \in [b,e] \text{ where } m \in \mathbb{N} \end{array} \right\}$$

Discrete- vs Continuous-Time DC

- The formula

$$\ell = 1 \iff \llbracket 1 \rrbracket \wedge \neg(\llbracket 1 \rrbracket \frown \llbracket 1 \rrbracket)$$

is valid for discrete time; but not for continuous time

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- The formula

  $$\llbracket S \rrbracket \Rightarrow (\llbracket S \rrbracket \frown \llbracket S \rrbracket)$$

  is valid for continuous time; but not for discrete time

Restricted Duration Calculus (*RDC*)

1. if S is a state expression, then $\llbracket S \rrbracket \in RDC$, and
2. if $\phi, \psi \in RDC$, then $\neg\phi, \phi \vee \psi, \phi \frown \psi \in RDC$.

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Expressiveness of *RDC* for Discrete Time:

$$
\begin{array}{lcl}
\ell = 0 & \iff & \neg[\![1]\!] \\
\int S = 0 & \iff & [\![\neg S]\!] \vee \ell = 0 \\
\ell = 1 & \iff & [\![1]\!] \wedge \neg([\![1]\!] \frown [\![1]\!]) \\
\int S = 1 & \iff & (\int S = 0) \frown ([\![S]\!] \wedge \ell = 1) \frown (\int S = 0) \\
\int S = k+1 & \iff & (\int S = k) \frown (\int S = 1) \\
\int S \geq k & \iff & (\int S = k) \frown \text{true} \\
\int S > k & \iff & (\int S \geq k) \wedge \neg(\int S = k) \\
\int S \leq k & \iff & \neg(\int S > k) \\
\int S < k & \iff & (\int S \leq k) \wedge \neg(\int S = k)
\end{array}
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Decidability of *RDC* for Discrete Time

Satisfiability is reduced to emptiness of regular languages

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Discrete time — one letter corresponds to one time unit

$$
\begin{aligned}
\mathcal{L}(\llbracket S \rrbracket) &= (DNF(S))^+ \\
\mathcal{L}(\varphi \vee \psi) &= \mathcal{L}(\varphi) \cup \mathcal{L}(\psi) \\
\mathcal{L}(\neg\varphi) &= \Sigma^* \setminus \mathcal{L}(\varphi) \\
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\end{array}$$

- $\mathcal{L}(\llbracket \phi \rrbracket)$ is regular
- ϕ is satisfiable iff $\mathcal{L}(\llbracket \phi \rrbracket \neq \emptyset$

Example

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\text{iff} & \neg((\llbracket P \rrbracket \frown \llbracket P \rrbracket) \Rightarrow \llbracket P \rrbracket) \text{ is not satisfiable} \\
\text{iff} & (\llbracket P \rrbracket \frown \llbracket P \rrbracket) \wedge \neg \llbracket P \rrbracket \text{ is not satisfiable} \\
\text{iff} & \mathcal{L}_1(\llbracket P \rrbracket \frown \llbracket P \rrbracket) \cap \mathcal{L}_1(\neg \llbracket P \rrbracket) = \{\} \\
\text{iff} & \{\{P\}^i \mid i \geq 2\} \cap (\Sigma^* \setminus \{\{P\}^i \mid i \geq 1\}) = \{\}
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The last equality holds.

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- Therefore, the formula is valid for discrete time.

Undecidability [Zhou Hansen Sestoft 93]

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Two-Counter Machines

- A *two-counter machine* has an *initial label* q_0, two *counters* c_1 and c_2 which can hold arbitrary natural numbers from $\mathbb{N} = \{0, 1, 2, \ldots\}$, and a finite set of labeled *instructions* m_i.

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- Instructions for c_1 (and similarly for c_2):

Instruction	s	$\Longrightarrow s'$
$q : c_1^+ \rightarrow q_j$	(q, n_1, n_2)	$\Longrightarrow (q_j, n_1 + 1, n_2)$
$q : c_1^- \rightarrow q_j, q_k$	$(q, 0, n_2)$	$\Longrightarrow (q_j, 0, n_2)$
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- Halting problem is undecidable

Assume deterministic machine with one halting state q_{fin}

Undecidability 1: Continuous time only

1. the formula $\ell = r$ belongs to $RDC_1(r)$,
2. if S is a state expression, then $\llbracket S \rrbracket$ belongs to $RDC_1(r)$, and
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Encoding of two-counter machine M:

- one state variable Q_i for each label q_i. Let $\mathcal{Q} = \{Q_0, \ldots, Q_{fin}\}$
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A configuration (q, n_1, n_2) is encoded on an interval of length $4r$:

$$| \underbrace{Q}_{r} | \underbrace{Val_1}_{r} | \underbrace{L}_{r} | \underbrace{Val_2}_{r} |$$

where Val_j represents the value of counter c_j.

Undecidability 1 – Abbreviations

$$
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\llbracket\ \rrbracket & \widehat{=} & \neg\llbracket 1 \rrbracket \\
\text{true} & \widehat{=} & \llbracket\ \rrbracket \vee \llbracket 1 \rrbracket \\
\ell < r & \widehat{=} & \neg((\ell = r)\frown\text{true}) \\
\ell = 4r & \widehat{=} & (\ell = 2r)\frown(\ell = 2r) \\
\llbracket S \rrbracket^r & \widehat{=} & \llbracket S \rrbracket \wedge (\ell = r) \\
\phi \leadsto \psi & \widehat{=} & \neg(\phi\frown\neg(\llbracket\ \rrbracket \vee \psi))
\end{array}
$$

- $\llbracket S \rrbracket^r$ reads "S has value one for a duration of r"
- $\phi \leadsto \psi$ reads "if the interval starts with ϕ, it must end immediately with $\llbracket\ \rrbracket$ or with ψ" — ϕ leads to ψ

Undecidability 1 – Continued

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$$|B|C_i|B|\cdots|B|C_i|B|$$

with n_i sections of C_i separated by B.

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The computation of M is simulated by a formula $F(M)$

For example, the following formula copies the C_1 sections to the same place in the next configuration.

$$\left(\llbracket Q_i \rrbracket^r \frown (\ell < r) \frown \llbracket C_1 \rrbracket \frown \left(\begin{array}{c} \llbracket C_1 \rrbracket \frown \textsf{true} \\ \wedge \\ \ell = 4r \end{array}\right)\right) \rightsquigarrow (\llbracket C_1 \rrbracket \frown \textsf{true})$$

exploits precision of length

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exploits precision of length

- M halts iff $F(M)$ is satisfiable
- satisfiability is undecidable for the subset under consideration

Remarks

- $\ell = 1$ is not expressible in RDC for continuous time.
- "Relaxing punctuality", replacing $\ell = r$ with $\ell < r$ does not give decidability.
- "Relaxing punctuality", replacing $\ell = r$ with $\ell > r$?

Undecidability 2: Discrete and Cont. Time

1. if S_1 and S_2 are state expressions, then $\int S_1 = \int S_2$ belongs to RDC_2, and
2. if ϕ and ψ belong to RDC_2, then so do $\neg\phi$, $\phi \vee \psi$ and $\psi \frown \psi$.

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Idea

- the value of c_i is represented by the value of $\int C_i^+ - \int C_i^-$
- a computation $s_0\ s_1\ s_2\ \cdots$
  is represented by a sequence $|QE_0|C_0|QE_1|C_1|QE_2|C_2|\cdots$
  - QE_k is a state expression of $\mathcal{Q}$
  - C_k is a state expression of $\{C_1^+, C_2^+, C_1, C_2\}$

Undecidability 2: Abbreviations

$$
\begin{array}{lcl}
C^{\vee} & \widehat{=} & C_1^+ \vee C_1^- \vee C_2^+ \vee C_2^- \\
\llbracket S \rrbracket & \widehat{=} & (\int S = \int 1) \wedge \neg(\int 0 = \int 1) \\
Incr_1 & \widehat{=} & \llbracket C_1^+ \wedge \neg(C_1^- \vee C_2^+ \vee C_2^-) \rrbracket \\
\int S > 0 & \widehat{=} & \Diamond \llbracket S \rrbracket \\
\llbracket\ \rrbracket & \widehat{=} & \neg \llbracket 1 \rrbracket \\
\textsf{true} & \widehat{=} & \llbracket\ \rrbracket \vee \llbracket 1 \rrbracket \\
\phi \leadsto \psi & \widehat{=} & \neg(\phi \frown \neg(\llbracket\ \rrbracket \vee \psi)) \\
\Diamond_p \phi & \widehat{=} & \phi \frown \textsf{true} \qquad \text{reads: “for some prefix interval: } \phi\text{”} \\
\Box_p \phi & \widehat{=} & \neg(\Diamond_p(\neg \phi)) \qquad \text{reads: “for all prefix intervals: } \phi\text{”.}
\end{array}
$$

Undecidability 2: Encoding Instructions

The instruction $q_j : c_i^+ \rightarrow q_k$ is encoded as follows

$$\left(\left(\left(\begin{array}{c} \llbracket\ \rrbracket \\ \vee \\ (\text{true} \frown \llbracket C^\vee \rrbracket) \end{array} \right) \frown \left(\begin{array}{c} (\llbracket Q_j \rrbracket \frown \llbracket C^\vee \rrbracket) \\ \wedge \\ \int Q_j = \int C^\vee \end{array} \right) \right) \begin{array}{c} \\ \\ \\ \rightsquigarrow \end{array} \left(\begin{array}{l} \llbracket Q_k \rrbracket \\ \vee \\ (\llbracket Q_k \rrbracket \frown Incr_i) \\ \vee \\ (\llbracket Q_k \rrbracket \frown Incr_i \frown \llbracket Q^\vee \rrbracket \frown \text{true}) \end{array} \right) \right)$$

- remaining instructions follows similar pattern
- mutual exclusive sections and sections of equal size
- undecidability of RDC_2

Undecidability 3

1. if S is a state expression, then $[\![S]\!]$ belongs to RDC_3,
2. if x is a global variable, then $\ell = x$ belongs to RDC_3, and
3. if ϕ and ψ belong to RDC_3, then so do $\neg\phi$, $\phi \vee \psi$, $\phi \frown \psi$ and $(\exists x)\phi$, where x is any global variable.

A configuration of the machine is represented by a sequence of sections Q, L and C, all of the same length:

$$|Q|\underbrace{C|\cdots|C}_{n_1}|L_1|\underbrace{C|\cdots|C}_{n_2}|L_2|$$

The initial configuration, $(q_0, 0, 0)$, is represented by $|Q_0|L_1|L_2|$:

$$\exists x.([\![Q_0]\!] \wedge (\ell = x)) \frown ([\![L_1]\!] \wedge (\ell = x)) \frown ([\![L_2]\!] \wedge (\ell = x)) \frown \text{true}$$

Undecidability 3: Encoding Instructions

An abbreviation:

$$[\![S]\!]^x \;\widehat{=}\; ([\![\;]\!] \vee [\![S]\!]) \wedge (\ell = x)$$

An instruction $q_j : c_1^+ \rightarrow q_k$ transforms configurations as follows:

$$|Q_j|\underbrace{C|\cdots|C}_{n_1}|L_1|\underbrace{C|\cdots|C}_{n_2}|L_2| \Longrightarrow |Q_k|\underbrace{C|C|\cdots|C}_{n_1+1}|L_1|\underbrace{C|\cdots|C}_{n_2}|L_2|$$

Encoding

$$\forall x, y, z.\left(\begin{array}{l}([\![Q_j]\!]^x \frown [\![C]\!]^y \frown [\![L_1]\!]^x \frown [\![C]\!]^z \frown [\![L_2]\!]^x \frown (\ell = 4x + y + z))\\ \Rightarrow \\ (\ell = 3x + y + z) \frown [\![Q_k]\!]^x \frown [\![C]\!]^x \frown [\![C]\!]^y \frown [\![L_1]\!]^x \frown [\![C]\!]^z \frown [\![L_2]\!]^x\end{array}\right.$$

- undecidability of RDC_3